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Recursion Formulae for Generalized Hypergeometric Functions[1]

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I. INTRODUCTION AND NOTATION

In Luke [*1*] and Fields [*2*], rational approximations to certain classes of hypergoemetric functions are developed. The results include as special cases the main and off diagonal entries of the Padé matrix [*3*, *4*] for the Gaussian hypergeometric function, one of whose numerator parameters is unity. A well-known property of this matrix is that the numerator and denominator of each entry satisfy the same three-term recurrence formula. Recently, Wimp [*5*] derived explicit recursion formulae for a certain class of hypergeometric functions closely related to the denominator polynomials of the Luke and Fields approximations. Thus, it is natural to ask, using a modified form of Wimp's analysis, whether the Luke and Fields approximations satisfy recurrence properties similar to those of the Padé matrix. This and related questions are answered in this paper.

The generalized hypergeometric function [*6*] is defined by the formal expression

$$
{}_pF_q(z) = {}_pF_q\left(\begin{matrix}\alpha_1, \ldots, \alpha_p\\ \beta_1, \ldots, \beta_q\end{matrix}\middle| z\right) = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p} (\alpha_j)_k}{\prod_{j=1}^{q} (\beta_j)_k} \cdot \frac{z^k}{k!}, \tag{1.1}
$$

where

$$
(\sigma)_\mu = \frac{\Gamma(\sigma+\mu)}{\Gamma(\sigma)}.
$$

We assume that no β_j is a nonpositive integer. For ease in writing, we employ the contracted notation

$$
{}_pF_q(z) = {}_pF_q\left(\begin{matrix}\alpha_p\\ \beta_q\end{matrix}\middle| z\right) = \sum_{k=0}^{\infty} \frac{(\alpha_p)_k}{(\beta_q)_k} \cdot \frac{z^k}{k!}. \tag{1.2}
$$

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Thus $(\alpha_p)_k$ is to be interpreted as $\prod_{j=1}^{p}(\alpha_j)_k$ and similarly for $(\beta_q)_k$. Similar notations such as $\Gamma(\alpha_p)$ standing for $\prod_{j=1}^{p}\Gamma(\alpha_j)$, and $(\alpha_p)^*_{-\alpha_t}$ standing for

$$\prod_{\substack{j=1\\ j\neq t}}^{p}(\alpha_j)_{-\alpha_t}$$

will be used throughout this paper. Considered as a power series in z, ${}_pF_q(z)$ has a radius of convergence equal to infinity if $p \leqslant q$, unity if $p = q+1$, and (in general) zero if $p \geqslant q+2$. If one of the α_j is a negative integer, the infinite series in (1.1) terminates. If no α_j is a negative integer, a meaning can still be given to ${}_pF_q(z)$, $p \geqslant q+2$, by considering it as the asymptotic expansion as $z \to 0$, of a certain type of contour integral.

More generally, we define Meijer's G-function [6] by

$$G_{p,q}^{m,n}\left(z\,\middle|\,\begin{matrix}a_p\\ \\ b_q\end{matrix}\right) = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j - s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j - s)}\, z^s\, ds, \tag{1.3}$$

where an empty product is interpreted as 1, $0 \leqslant m \leqslant q$, $0 \leqslant n \leqslant p$, the parameters are such that no pole of $\Gamma(b_j - s), j = 1,\ldots, m$ coincides with any pole of $\Gamma(1 - a_k + s)$, $k = 1,\ldots, n$, and where the path L runs parallel to the imaginary axis, and is indented to separate the poles of $\Gamma(b_m - s)$ from the poles of $\Gamma(1 - a_n + s)$. The above integral is well defined if $p + q < 2(m + n)$ and $|\arg z| < [(m + n) - (p + q)/2]\pi$. If all the poles of the integrand in (1.3) are simple, it is easy to see from the residue theorem, that

$$G_{p,q}^{m,n}\left(z\,\middle|\,\begin{matrix}a_p\\ \\ b_q\end{matrix}\right)$$

can be represented as a sum of well-defined hypergeometric functions, e.g.,

$$G_{p,q}^{m,n}\left(z\,\middle|\,\begin{matrix}a_p\\ \\ b_q\end{matrix}\right)$$

$$= \sum_{h=1}^{m} \frac{\prod_{\substack{j=1\\ j\neq h}}^{m}\Gamma(b_j - b_h)\prod_{j=1}^{n}\Gamma(1+b_h-a_j)\, z^{b_h}}{\prod_{j=m+1}^{q}\Gamma(1+b_h-b_j)\prod_{j=n+1}^{p}\Gamma(a_j - b_h)}\, {}_{p+1}F_q\left(\begin{matrix}1, 1+b_h-a_p\\ \\ 1+b_h-b_q\end{matrix}\,\middle|\,(-1)^{p-m-n}z\right),$$

$$p < q \quad\text{or}\quad p = q \quad\text{and}\quad |z| < 1. \tag{1.4}$$

A similar expansion holds if $p > q$ or $p = q$ and $|z| > 1$, and follows directly from (1.4) and the functional relationship

$$G_{p,q}^{m,n}\left(z^{-1}\left|\begin{matrix}a_p\\ \\ b_q\end{matrix}\right.\right) = G_{q,p}^{n,m}\left(z\left|\begin{matrix}1-b_q\\ \\ 1-a_p\end{matrix}\right.\right). \tag{1.5}$$

Both functional relationships (1.5) and

$$z^c\, G_{p,q}^{m,n}\left(z\left|\begin{matrix}a_p\\ \\ b_q\end{matrix}\right.\right) = G_{p,q}^{m,n}\left(z\left|\begin{matrix}c+a_p\\ \\ c+b_q\end{matrix}\right.\right), \tag{1.6}$$

follow directly from the integral definition (1.3).

A special case of (1.4) is

$$E_{p,q}(z) \equiv G_{p,q+1}^{1,p}\left(-z\left|\begin{matrix}1-\alpha_p\\ \\ 0,1-\beta_q\end{matrix}\right.\right)$$

$$= \frac{\Gamma(\alpha_p)}{\Gamma(\beta_q)}\,{}_pF_q\left(\left.\begin{matrix}\alpha_p\\ \\ \beta_q\end{matrix}\right|z\right); \qquad p < q+1 \quad \text{or} \quad p = q+1, \quad |\arg(-z)| < \pi. \tag{1.7}$$

Thus, $E_{q+1,q}(z)$ analytically extends ${}_{q+1}F_q(z)$ into the region $|\arg(1-z)| < \pi$. Moreover, it can be shown [7] that for $p \geqslant q+2$,

$$E_{p,q}(z) \sim \frac{\Gamma(\alpha_p)}{\Gamma(\beta_q)}\,{}_pF_q\left(\left.\begin{matrix}\alpha_p\\ \\ \beta_q\end{matrix}\right|z\right),$$

$$|\arg(-z)| < (p+1-q)\pi/2, \qquad z \to 0. \tag{1.8}$$

The formal Luke and Fields rational approximations to $E_{p,q}(z)$, $\psi_n(z,\gamma)/f_n(\gamma)$, are defined as follows. For $a = 0$ or 1, and the parameters $A_{n,k}$, γ arbitrary, set

$$f_n^{[r]}(\gamma) = \sum_{k=r}^{n} A_{n,k}\gamma^k; \qquad f_n(\gamma) = f_n^{[0]}(\gamma), \tag{1.9}$$

$$\psi_n(z,\gamma) = \sum_{k=0}^{n} A_{n,k}\gamma^k P_{k-a+1}(z),$$

$$= \sum_{r=0}^{n-a} \frac{\Gamma(r+\alpha_p)}{\Gamma(r+\beta_q)\,r!} z^r f_n^{[r+a]}(\gamma),$$

$$= \sum_{k=a}^{n} z^{-k} \sum_{r=k}^{n} \frac{A_{n,r}\Gamma(r-k+\alpha_p)}{\Gamma(r-k+\beta_q)(r-k)!}(\gamma z)^r, \tag{1.10}$$

$$P_k(z) = \sum_{j=0}^{k-1} \frac{\Gamma(j+\alpha_p)}{\Gamma(j+\beta_q)\, j!} z^j, \qquad P_0(z) = 0. \tag{1.11}$$

It was shown in [2] that if

$$A_{n,k} = \frac{(-n)_k (n+\lambda)_k (\beta_q - a)_k}{(\beta+1)_k (\alpha_p + 1 - a)_k}; \qquad \lambda, \beta \text{ arbitrary}, \tag{1.12}$$

then

$$E_{p,q}(z) = \lim_{n \to +\infty} \frac{\psi_n(z,\gamma)}{f_n(\gamma)}, \tag{1.13}$$

under quite general restrictions on p, q, z, γ, etc. The significance of the parameter a is plainly seen, if in the last line of (1.10) one successively sets $z = \gamma^{-1}$ and $\gamma = 0$. For then $\psi_n(\infty, 0)$ equals zero if $a = 1$, and is not equal to zero if $a = 0$. Classically, the cases $a = 0$ and $a = 1$ correspond to taking the odd and even convergents, respectively, of certain continued fractions, see [6, 8].

We note that if $A_{n,k}$ is chosen as in (1.12), the denominator polynomial $f_n(\gamma)$ is of the general hypergeometric form

$${}_{r+2}F_s\left(\begin{matrix} -n, n+\lambda, \alpha_r \\ \beta_s \end{matrix}\middle| z\right) \tag{1.14}$$

which is known as the extended Jacobi polynomial. A limiting form of the extended Jacobi polynomial is the extended Laguerre polynomial

$${}_{r+1}F_s\left(\begin{matrix} -n, \alpha_r \\ \beta_s \end{matrix}\middle| z\right). \tag{1.15}$$

If n is not an integer, (1.14) and (1.15) are known as extended Jacobi and Laguerre functions, respectively. In Section II, explicit linear recursion equations for such polynomials (functions) are derived. In Section III, linear recursion equations for the corresponding numerator polynomials, $\psi_n(z, \gamma)$, are also derived. Our results are stated quite generally.

In Section IV, we relate the material of the previous sections to the problem of finding recursion relationships for the coefficients in the expansion of Meijer G-functions in series of extended Jacobi and Laguerre polynomials. In particular, it is shown in [9, 10, 11], that under sufficient restrictions,

$$G_{p+r,q+s}^{m,k+r}\left(\omega z\middle|\begin{matrix} c_r, a_p \\ b_q, d_s \end{matrix}\right) = \frac{\Gamma(1-c_r)}{\Gamma(1-d_s)} \sum_{n=0}^{\infty} \frac{(-)^n (2n+\lambda)\, \Gamma(n+\lambda)}{n!}$$
$$\times\, G_{p+1,q+2}^{m,k+1}\left(\omega\middle|\begin{matrix} 0, a_p \\ b_q, n, -n-\lambda \end{matrix}\right) {}_{r+2}F_s\left(\begin{matrix} -n, n+\lambda, 1-c_r \\ 1-d_s \end{matrix}\middle| z\right), \tag{1.16}$$

$$G^{m,k+r}_{p+r,q+s}\left(\omega z\,\middle|\,\begin{matrix}c_r, a_p\\ \\ b_q, d_s\end{matrix}\right)=\frac{\Gamma(1-c_r)}{\Gamma(1-d_s)}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}$$

$$\times\, G^{m,k+1}_{p+1,q+1}\left(\omega\,\middle|\,\begin{matrix}0, a_p\\ \\ b_q, n\end{matrix}\right){}_{r+1}F_s\left(\begin{matrix}-n, 1-c_r\\ \\ 1-d_s\end{matrix}\,\middle|\,z\right). \tag{1.17}$$

For generalized hypergeometric functions, (1.16) and (1.17) can be interpreted as

$${}_{p+r}F_{q+s}\left(\begin{matrix}\alpha_p, \sigma_r\\ \\ \beta_q, \rho_s\end{matrix}\,\middle|\,\omega z\right)=\sum_{n=0}^{\infty}\frac{(\alpha_p)_n\,(-\omega)^n}{(\beta_q)_n(n+\lambda)_n\,n!}$$

$$\times\,{}_pF_{q+1}\left(\begin{matrix}n+\alpha_p\\ \\ n+\beta_q, 2n+\lambda+1\end{matrix}\,\middle|\,\omega\right){}_{r+2}F_s\left(\begin{matrix}-n, n+\lambda, \sigma_r\\ \\ \rho_s\end{matrix}\,\middle|\,z\right), \tag{1.18}$$

and

$${}_{p+r}F_{q+s}\left(\begin{matrix}\alpha_p, \sigma_r\\ \\ \beta_q, \rho_s\end{matrix}\,\middle|\,\omega z\right)=\sum_{n=0}^{\infty}\frac{(\alpha_p)_n(-\omega)^n}{(\beta_q)_n\,n!}$$

$$\times\,{}_pF_q\left(\begin{matrix}n+\alpha_p\\ \\ n+\beta_q\end{matrix}\,\middle|\,\omega\right){}_{r+1}F_s\left(\begin{matrix}-n, \sigma_r\\ \\ \rho_s\end{matrix}\,\middle|\,z\right). \tag{1.19}$$

II. Recursion Formulae for the Extended Jacobi and Laguerre Functions

In the following, we shall derive a linear, nonhomogeneous difference equation for the generalized Jacobi function,

$$\mathscr{E}_n(z,\lambda)={}_{r+3}F_s\left(\begin{matrix}-n, n+\lambda, \alpha_r, 1\\ \\ \beta_s\end{matrix}\,\middle|\,z\right);\qquad n \text{ arbitrary},$$

$$r+3\leqslant s,\quad\text{or}\quad r+2=s\quad\text{and}\quad|\arg(1-z)|<\pi. \tag{2.1}$$

Complementary to $\mathscr{E}_n(z,\lambda)$ is the function

$$\mathscr{K}_n(z,\lambda)=\frac{(\beta_s-1)\,z^{-1}}{(n+1)(n+\lambda-1)(\alpha_r-1)}$$

$$\times\,{}_{s+1}F_{r+2}\left(\begin{matrix}2-\beta_s, 1\\ \\ 2+n, 2-n-\lambda, 2-\alpha_r\end{matrix}\,\middle|\,\frac{(-1)^{s-r}}{z}\right),$$

$$r+1\geqslant s,\quad\text{or}\quad r+2=s\quad\text{and}\quad|\arg(1-1/z)|<\pi, \tag{2.2}$$

in the sense that both are particular solutions of the differential equation

$$[(\delta+\beta_s-1)-z(\delta-n)(\delta+n+\lambda)(\delta+\alpha_r)]\,Y(z)=(\beta_s-1), \tag{2.3}$$

where

$$(\delta+\alpha_r)=\prod_{j=1}^{r}(\delta+\alpha_j),\ \text{etc.,} \qquad \delta=z\frac{d}{dz}. \tag{2.4}$$

We shall not only show that $\mathscr{E}_n(z,\lambda)$ and $\mathscr{K}_n(z,\lambda)$ satisfy the same linear, non-homogeneous difference equation, but that a properly normalized basis of the related, homogeneous differential equation

$$[(\delta+\beta_s-1)-z(\delta-n)(\delta+n+\lambda)(\delta+\alpha_r)]\,Y(z)=0, \tag{2.5}$$

also satisfies the related, homogeneous difference equation.

To describe these bases, normalized with respect to n, it is convenient to write down the following sets of conditions:

$$\left.\begin{array}{l} r+3\leqslant s, \quad \text{or} \quad r+2=s \quad \text{and} \quad |\arg(1-z)|<\pi, \\ \text{no two of the parameters, } \beta_h (h=1,\ldots,s), \text{ differ by an} \\ \text{integer,} \end{array}\right\} C_{0,\lambda}$$

$$\left.\begin{array}{l} r+1\geqslant s, \quad \text{or} \quad r+2=s \quad \text{and} \quad |\arg(1-1/z)|<\pi, \\ \text{no two of the parameters, } -n, n+\lambda, \alpha_k (k=1,\ldots,r), \text{ differ} \\ \text{by an integer.} \end{array}\right\} C_{\infty,\lambda}$$

Under condition $C_{0,\lambda}$, we take for our normalized basis,

$$\begin{aligned} &\mathscr{F}_{n,h}(z,\lambda) \\ &= (n+\beta_h)_{1-\beta_h}(n+\lambda)_{1-\beta_h}\, z^{1-\beta_h} \\ &\quad\times {}_{r+3}F_s\left(\begin{matrix}1,\ 1-\beta_h-n,\ 1-\beta_h+n+\lambda,\ 1-\beta_h+\alpha_r \\ 1-\beta_h+\beta_s\end{matrix}\,\middle|\, z\right) \\ &\qquad h=1,\ldots,s. \end{aligned} \tag{2.6}$$

Alternatively, under condition $C_{\infty,\lambda}$, we take for our normalized basis,

$$\begin{aligned} &\mathscr{G}_{n,k}(z,\lambda) \\ &= (n+1+\alpha_k)_{-\alpha_k}(n+\lambda)_{-\alpha_k}\, z^{-\alpha_k} \\ &\quad\times {}_{s+1}F_{r+2}\left(\begin{matrix}1,\ 1+\alpha_k-\beta_s \\ 1+\alpha_k+n,\ 1+\alpha_k-n-\lambda,\ 1+\alpha_k-\alpha_r\end{matrix}\,\middle|\, \frac{(-1)^{s-r}}{z}\right), \\ &\qquad k=1,\ldots,r. \end{aligned} \tag{2.7}$$

$$\mathscr{G}_{n,\,r+1}(z,\lambda)$$
$$=\frac{\Gamma(n+1)\,\Gamma(2n+\lambda)\,\Gamma(n+\alpha_r)\,e^{in\phi}\,z^n}{\Gamma(n+\lambda)\,\Gamma(n+\beta_s)}$$
$$\times\,{}_sF_{r+1}\left(\begin{matrix}1-\beta_s-n\\ 1-2n-\lambda,1-\alpha_r-n\end{matrix}\,\middle|\,\frac{(-1)^{s-r}}{z}\right), \tag{2.8}$$

$$\mathscr{G}_{n,\,r+2}(z,\lambda)$$
$$=\frac{\Gamma(n+1)\,\Gamma(n+\lambda+1-\beta_s)\,e^{in\phi(s-r-1)}\,z^{-n-\lambda}}{\Gamma(n+\lambda)\,\Gamma(2n+\lambda+1)\,\Gamma(n+\lambda+1-\alpha_r)}$$
$$\times\,{}_sF_{r+1}\left(\begin{matrix}n+\lambda+1-\beta_s\\ 2n+\lambda+1,n+\lambda+1-\alpha_r\end{matrix}\,\middle|\,\frac{(-1)^{s-r}}{z}\right), \tag{2.9}$$

where $e^{i\phi}=-1$.

With these definitions, we state

THEOREM 2.1. *The functions $\mathscr{E}_n(z,\gamma)$ and $\mathscr{K}_n(z,\gamma)$ under the conditions on r, s and z in $C_{0,\lambda}$ and $C_{\infty,\lambda}$, respectively, satisfy the difference equation*

$$\Phi_n(z,\lambda)+\sum_{m=1}^{t}\,[A_m(n,\lambda)+zB_m(n,\lambda)]\,\Phi_{n-m}(z,\lambda)=\frac{(\beta_s-1)(n+\lambda)_n}{(n+\beta_s-1)(n+\lambda-t)_n},$$
$$t=\max\,(r+2,s),\quad B_t(n,\lambda)=0 \tag{2.10}$$

where

$$A_m(n,\lambda)$$
$$=\frac{(n+1-m)_m(2n+\lambda-2m)_{2m}(n-m-1+\beta_s)}{m!\,(n+\lambda-m)_m(2n+\lambda-t-m)_m(n-1+\beta_s)}$$
$$\times\,{}_{s+2}F_{s+1}\left(\begin{matrix}-m,2n+\lambda-t-m,n-m+\beta_s\\ 2n+\lambda+1-2m,n-m-1+\beta_s\end{matrix}\,\middle|\,1\right),$$
$$=\frac{(-)^s(n+1-m)_m(2n+\lambda-2m)(2n+\lambda-t+1)_{t-1}(n+\lambda-t+1-\beta_s)}{(t-m)!\,(n+\lambda-m)_m(2n+\lambda-t-m)_m(n-1+\beta_s)}$$
$$\times\,{}_{s+2}F_{s+1}\left(\begin{matrix}-t+m,2n+\lambda-t-m,n+\lambda-t+2-\beta_s\\ 2n+\lambda+1-t,n+\lambda-t+1-\beta_s\end{matrix}\,\middle|\,1\right), \tag{2.11}$$

$$
\begin{aligned}
&B_m(n,\lambda)\\
&= \frac{(n+1-m)_m(2n+\lambda-2m)_{2m}(n-m+\alpha_r)}{(m-1)!\,(n+\lambda-m)_m(2n+\lambda-t-m+1)_{m-1}(n-1+\beta_s)}\\
&\quad\times {}_{r+2}F_{r+1}\left(\begin{matrix}1-m, 2n+\lambda-t-m+1, n-m+1+\alpha_r\\ 2n+\lambda+1-2m, n-m+\alpha_r\end{matrix}\middle| 1\right),\\
&= \frac{(-)^r(n+1-m)_m(2n+\lambda-2m)(2n+\lambda-t+1)_{t-1}(n+\lambda-t+1-\alpha_r)}{(t-m-1)!\,(n+\lambda-m)_m(2n+\lambda-t-m+1)_{m-1}(n-1+\beta_s)}\\
&\quad\times {}_{r+2}F_{r+1}\left(\begin{matrix}-t+m+1, 2n+\lambda-t-m+1, n+\lambda-t+2-\alpha_r\\ 2n+\lambda+1-t, n+\lambda-t+1-\alpha_r\end{matrix}\middle| 1\right).
\end{aligned}
\tag{2.12}
$$

In addition, the functions $\mathscr{F}_{n,h}(z,\lambda)$ $(h=1,\ldots,s)$ and $\mathscr{G}_{n,k}(z,\lambda)$ $(k=1,\ldots,r+2)$ under the conditions $C_{0,\lambda}$ and $C_{\infty,\lambda}$, respectively, satisfy the difference equation

$$\Phi_n(z,\lambda)+\sum_{m=1}^{t}[A_m(n,\lambda)+zB_m(n,\lambda)]\Phi_{n-m}(z,\lambda)=0. \tag{2.13}$$

Finally, if no α_k is equal to any β_h, none of the above functions satisfy a nontrivial equation of the form specified of lower order than t.

Proof. By analytic continuation with respect to z, it is sufficient to prove the theorem with the conditions $C_{0,\lambda}$ and $C_{\infty,\lambda}$ strengthened to $|z|<1$ and $|z|>1$, respectively. Tentatively, we assume that no α_k is equal to any β_h, and that we wish to prove

$$\sum_{m=0}^{t}[A_m(n,\lambda)+zB_m(n,\lambda)]\Phi_{n-m}(z,\lambda)=K(\Phi_n(z,\lambda)),\qquad A_0(n,\lambda)=1, \tag{2.14}$$

where $K(\Phi_n(z,\lambda))$ is a monomial in z, which depends upon the identity of $\Phi_n(z,\lambda)$. A necessary and sufficient condition that $\mathscr{E}_n(z,\lambda)$, $\mathscr{K}_n(z,\lambda)$, $\mathscr{F}_{n,h}(z,\lambda)$ or $\mathscr{G}_{n,k}(z,\lambda)$ satisfy (2.14), is that when these functions are substituted in (2.14), and the resulting equations are rearranged in powers of z, the coefficients of z^j, z^{-j}, $z^{1-\beta_h+j}$ and $z^{1-\sigma_k-j}$ $(\sigma_k=\alpha_k$ $(k=1,\ldots,r)$, $\sigma_{r+1}=-n$, $\sigma_{r+2}=n+\lambda-t)$, respectively, are zero for $j=1,2,\ldots$, while the terms corresponding to $j=0$ reduce to $K(\Phi_n(z,\lambda))$. In order to state these conditions explicitly, we introduce the polynomials

$$X_t(w)=\sum_{m=0}^{t}\frac{(w-n)_m(w+n+\lambda-t)_{t-m}}{(-n)_m(n+\lambda-t)_{t-m}}A_m(n,\lambda),$$

$$Y_t(w)=-\sum_{m=0}^{t}\frac{(w-n-1)_m(w+n+\lambda-t-1)_{t-m}}{(-n)_m(n+\lambda-t)_{t-m}}B_m(n,\lambda). \tag{2.15}$$

Then, after some algebra, the above conditions can be written in the form

$$K(\mathscr{E}_n(z,\lambda)) = X_t(0),$$

$$K(\mathscr{K}_n(z,\lambda)) = -\frac{(\beta_s - 1)\, Y_t(0)}{(\alpha_r - 1)(n+1)(n+\lambda-t-1)},$$

$$K(\mathscr{F}_{n,h}(z,\lambda))$$
$$= (n+\beta_h)_{1-\beta_h}(n+\lambda-t)_{1-\beta_h}\, X_t(1-\beta_h)\, z^{1-\beta_h}, \qquad h = 1, \ldots, s,$$

$$K(\mathscr{G}_{n,k}(z,\lambda))$$
$$= -(n+1+\alpha_k)_{-\alpha_k}(n+\lambda-t)_{-\alpha_k}\, Y_t(1-\alpha_k)\, z^{1-\alpha_k}, \qquad k = 1, \ldots, r,$$

$$K(\mathscr{G}_{n,r+1}(z,\lambda))$$
$$= \frac{\Gamma(n+1)\,\Gamma(2n+\lambda-t)\,\Gamma(n+\alpha_r)\, e^{in\phi}}{\Gamma(n+\lambda-t)\,\Gamma(n+\beta_s)}\, Y_t(1+n)\, z^{1+n},$$

$$K(\mathscr{G}_{n,r+2}(z,\lambda))$$
$$= \frac{-\Gamma(n+1)\,\Gamma(n+\lambda-t+1-\beta_s)\, e^{i\phi(s-r-1)(n-t)}}{\Gamma(n+\lambda-t)\,\Gamma(2n+\lambda-t+1)\,\Gamma(n+\lambda-t+1-\alpha_r)}$$
$$\times\, Y_t(1-n-\lambda+t)\, z^{1-n-\lambda+t}, \tag{2.16}$$

and

$$X_t(w)(w-n-1)(w+n+\lambda-t-1)\prod_{k=1}^{r}(w+\alpha_k-1) = Y_t(w)\prod_{h=1}^{s}(w+\beta_h-1), \tag{2.17}$$

whenever $w = j,\ -j,\ 1-\beta_h+j$ or $1-\sigma_k-j$, for $\mathscr{E}_n(z,\lambda)$, $\mathscr{K}_n(z,\lambda)$, $\mathscr{F}_{n,h}(z,\lambda)$, or $\mathscr{G}_{n,k}(z,\lambda)$, respectively.

As (2.17) can be viewed as a polynomial form in w of degree $\leqslant 2t$, (2.17) must actually hold for all w. Moreover, as the α_k and β_h are independent of n, and assumed unequal,

$$\prod_{h=1}^{s}(w+\beta_h-1)\left((w-n-1)(w+n+\lambda-t-1)\prod_{k=1}^{r}(w+\alpha_k-1)\right)$$

must divide $X_t(w)$ $(Y_t(w))$, and the resulting polynomial will have degree $\leqslant \max(r+2-s, 0)$ $(\leqslant \max(0, s-r-2))$. Suppose $r+2 \leqslant s$. Then there exists a number C independent of w such that

$$X_t(w) = C\prod_{h=1}^{s}(w+\beta_h-1), \tag{2.18}$$

and substitution of this identity into (2.17) yields

$$Y_t(w) = C(w - n - 1)(w + n + \lambda - t - 1) \prod_{k=1}^{r} (w + \alpha_k - 1). \tag{2.19}$$

The alternate assumption, $r + 2 \geqslant s$, again leads to (2.18) and (2.19). The value of the constant C follows from $X_t(n)$, i.e.,

$$C = \frac{X_t(n)}{(n + \beta_s - 1)} = \frac{(2n + \lambda - t)_t A_0(n, \lambda)}{(n + \lambda - t)_t (n + \beta_s - 1)} = \frac{(n + \lambda)_n}{(n + \lambda - t)_n (n + \beta_s - 1)}. \tag{2.20}$$

Moreover, it follows from (2.16), (2.18) and (2.19), that

$$\begin{aligned} K(\mathscr{E}_n(z, \lambda)) &= K(\mathscr{K}_n(z, \lambda)) = \frac{(\beta_s - 1)(n + \lambda)_n}{(n + \beta_s - 1)(n + \lambda - t)_n}, \\ K(\mathscr{F}_{n,h}(z, \lambda)) &= K(\mathscr{G}_{n,k}(z, \lambda)) = 0. \end{aligned} \tag{2.21}$$

Finally, the values of $A_m(n, \lambda)$ ($B_m(n, \lambda)$) given in (2.11) ((2.12)) follow from (2.18) ((2.19)) by an application of the following lemma.

Lemma 2.1. *If $P_q(x)$ is a polynomial in x of degree q,*

$$P_q(x) = c \prod_{i=1}^{q} (x - \omega_i), \tag{2.22}$$

and t is an integer $\geqslant q$, then $P_q(x)$ can be represented uniquely in the form

$$P_q(x) = \sum_{m=0}^{t} (x + \gamma)_m (x + \gamma + \epsilon)_{t-m} Q_m, \tag{2.23}$$

$$\begin{aligned} Q_m &= \frac{(-)^q (t + \epsilon - 2m)(\gamma + \omega_q) c}{m! \, (\epsilon)_{t+1-m}} \\ &\quad \times {}_{q+2}F_{q+1}\left(\left. \begin{matrix} -m, m - \epsilon - t, 1 + \gamma + \omega_q \\ 1 - \epsilon, \gamma + \omega_q \end{matrix} \right| 1 \right), \\ &= \frac{(-1)^{q+t} (\gamma + \epsilon + t - m + \omega_q) c}{(t - m)! \, (1 + \epsilon + t - 2m)_m} \\ &\quad \times {}_{q+2}F_{q+1}\left(\left. \begin{matrix} m - t, m - t - \epsilon, 1 + m - t - \gamma - \epsilon - \omega_q \\ 1 + 2m - t - \epsilon, m - t - \gamma - \epsilon - \omega_q \end{matrix} \right| 1 \right), \end{aligned} \tag{2.24}$$

provided $\epsilon \neq 0, \pm 1, \ldots, \pm(t - 1)$. Note that $Q_m = 0$ if $m \geqslant t + 1$. If $(\gamma + \omega_q)$ or $(\gamma + \epsilon + \omega_q)$ are zero, limits must be taken in (2.24).

Proof. We first show that given $P_q(x)$, the coefficients Q_m, if they exist at all, are unique. Suppose there exists a set of $Q_m{}^*$'s which also satisfy (2.23). Then by subtraction

$$\sum_{m=0}^{t} (x+\gamma)_m(x+\gamma+\epsilon)_{t-m}(Q_m - Q_m{}^*) = 0 \qquad (2.25)$$

for all x. In (2.25) put $x+\gamma=-j$, $j=0, 1, \ldots, t$. Then by Cramer's rule, $Q_m = Q_m{}^*$ for each $m=0, 1, \ldots, t$ provided that Δ, the determinant of the coefficients of $(Q_m - Q_m{}^*)$ in the system derived from (2.25), does not vanish. Clearly, Δ is a lower triangular determinant and is simply evaluated as the product of all the elements on its main diagonal. Thus,

$$\Delta = \prod_{m=0}^{t} (-)^m m!\,(\epsilon - m)_{t-m} \neq 0$$

under the conditions on ϵ given after (2.24).

Consider the representation formula

$$P_q(x) = \sum_{k=0}^{t} \frac{(-)^k(x+\gamma)_t(x+\gamma+t)}{k!\,(t-k)!\,(x+\gamma+k)} P_q(-\gamma-k), \qquad (2.26)$$

which can be proved as follows: Since

$$\frac{(x+\gamma)_t(x+\gamma+t)}{x+\gamma+k} = (x+\gamma)_k(x+\gamma+k+1)_{t-k},$$

the right-hand side of (2.26) is a polynomial in x of degree t. By direct computation, the right-hand side of (2.26) agrees with $P_q(x)$ at the $t+1$ points $x=-\gamma-r$, $r=0, 1, \ldots, t$. Since $q \leqslant t$, this is sufficient to establish (2.26). To derive (2.23), we use Lemma A.1 proved in the Appendix with $n=t-k$, $\beta=(\epsilon-t)/2$ and $z=x+\gamma+t$. Thus

$$\begin{aligned} &{}_4F_3\left(\begin{matrix} k-t, x+\gamma+\epsilon, 1+\dfrac{\epsilon-t}{2}, 1 \\ \\ 1+\epsilon-k, 1-x-\gamma-t, \dfrac{\epsilon-t}{2} \end{matrix}\middle| 1\right) \\ &= \frac{(\epsilon-k)(x+\gamma+t)}{(\epsilon-t)(x+\gamma+k)}, \qquad (\epsilon-t)(x+\gamma+k)\neq 0. \end{aligned} \qquad (2.27)$$

Now put $(x+\gamma+t)/(x+\gamma+k)$ from the latter formula into (2.26), express the ${}_4F_3$ as a sum over m from 0 to t, interchange summation processes and so obtain (2.23) with

$$Q_m = \frac{(t+\epsilon-2m)}{m!\,(\epsilon)_{t+1-m}} \sum_{k=0}^{m} \frac{(-m)_k(m-\epsilon-t)_k}{k!(1-\epsilon)_k} P_q(-\gamma-k). \qquad (2.28)$$

Since

$$P_q(-\gamma-k) = (-)^q c \prod_{j=1}^{q} \frac{(\gamma+\omega_j)(1+\gamma+\omega_j)_k}{(\gamma+\omega_j)_k}, \tag{2.29}$$

(2.28) and (2.29) reduce to the first line of (2.24). Finally, to get the second line of (2.24), observe that

$$P_q(x) = \sum_{m=0}^{t} (x+\gamma^*)_m (x+\gamma^*+\epsilon^*)_{t-m}\, Q_m^*,$$

$$\gamma^* = \gamma+\epsilon, \quad \epsilon^* = -\epsilon, \quad Q_m^* = Q_{t-m}, \tag{2.30}$$

or

$$Q_m = \frac{(-1)^q(2m-\epsilon-t)(\gamma+\epsilon+\omega_q)\,c}{(t-m)!\,(-\epsilon)_{m+1}}$$

$$\times\, {}_{q+2}F_{q+1}\left(\begin{matrix} m-t, -m+\epsilon, 1+\gamma+\epsilon+\omega_q \\ \\ 1+\epsilon, \gamma+\epsilon+\omega_q \end{matrix}\middle| 1\right).$$

Turning this last series expression for Q_m around, we arrive at the second line of (2.24), which completes the proof of the lemma.

Under our tentative assumption that no α_k is equal to any β_h, the preceding lemma determines the $A_m(n,\lambda)$'s and $B_m(n,\lambda)$'s uniquely. This is sufficient to establish the last statement of the theorem. To see this explicitly, we note that $A_t(n,\lambda) \neq 0$ and assume that one of the functions of interest, $\Phi_n(z,\lambda)$, satisfies a difference equation of lower order than that specified, i.e.,

$$\sum_{m=0}^{t'} [A_m'(n,\lambda) + zB_m'(n,\lambda)]\,\Phi_{n-m}(z,\lambda) = R_n(z,\lambda), \qquad A_0'(n,\lambda) = 1 \tag{2.31}$$

but with $t' < t$. If in this assumed equation, (2.31), we replace n by $n-1$, multiply the resulting equation by an arbitrary parameter ρ_1 and add it to the original equation, (2.31), we would then obtain an equation of the form (2.31), but with t' replaced by $t'+1$. After $t-t'$ repetitions of this process, the resulting equation would still be of the form (2.31), but with t' replaced by t. Since, by the lemma, the $A_m(n,\lambda)$'s are unique, we would have, in particular,

$$A_t(n,\lambda) = \rho_{t-t'} A_{t'}'(n-t,\lambda); \qquad \rho_{t-t'} \text{ arbitrary},$$

which would contradict the nonzero uniqueness of $A_t(n,\lambda)$. Finally, our tentative assumption that no α_k is equal to any β_h can be relaxed completely by an appeal to continuity. The only penalty exacted for such a relaxation is that the $A_m(n,\lambda)$'s and $B_m(n,\lambda)$'s are no longer unique, and that the recurrence formulae are no longer of the lowest possible order. This completes the proof of the theorem.

COROLLARY 2.1. *The function*

$$\mathscr{E}(n,z,\lambda)=\frac{\Gamma(\beta_s)}{\Gamma(-n)\,\Gamma(n+\lambda)\,\Gamma(\alpha_r)}\,G^{1,r+3}_{r+3,s+1}\left(-z\,\middle|\,\begin{matrix}0,1+n,1-n-\lambda,1-\alpha_r\\ 0,1-\beta_s\end{matrix}\right) \tag{2.32}$$

satisfies the difference equation (2.10), *and if no* α_k *is equal to any* β_h, *satisfies no nontrivial equation of the same form, of lower order than* t.

Proof. Under conditions $C_{0,\lambda}$, $\mathscr{E}(n,z,\lambda)=\mathscr{E}_n(z,\lambda)$, while under conditions $C_{\infty,\lambda}$, it follows from (1.4) that $\mathscr{E}(n,z,\lambda)$ differs from $\mathscr{K}_n(z,\lambda)$ by a linear combination of the functions $\mathscr{G}_{n,k}(z,\lambda)$ whose coefficients are independent of both z and n, except possibly for a periodic function of n which has period unity.

Remark 2.1. The determination of the $A_m(n,\lambda)$'s and $B_m(n,\lambda)$'s was first given by Wimp [5] in the special case that one of the β_h's is unity. He gives two proofs. One of these is algebraic and essentially stems from the solution of the linear equation systems derived from (2.18) and (2.19) when one puts therein $w=1-\beta_h$, $h=1,\ \ldots,\ s$ and $w=1+n,\ 1-n-\lambda+t,\ 1-\alpha_k,\ k=1,\ \ldots,\ r$, respectively. The other proof shows that if the $A_m(n,\lambda)$'s and $B_m(n,\lambda)$'s are as given, they can be represented by contour integrals and that $\mathscr{E}_n(z,\lambda)$ with one of the β_h's equal to unity satisfies the then homogeneous difference equation (2.10).

Remark 2.2. The generalized Jacobi function $\mathscr{E}_n(z,\lambda)$ will lose its specialized appearance, if we let $s=q+1$ and set $\beta_h=1, h=q+1$.

Remark 2.3. As previously noted, under conditions $C_{0,\lambda}$ ($C_{\infty,\lambda}$), the functions $\mathscr{F}_{n,h}(z,\lambda)$ ($\mathscr{G}_{n,k}(z,\lambda)$), form a basis of the differential equation (2.5), and hence are linearly independent as functions of z. Although we have shown that these same functions satisfy the difference equation (2.13), it is not known whether they form a basis of solutions of (2.13), i.e., whether they are linearly independent as functions of n.

Remark 2.4. If the parameter condition in $C_{0,\lambda}$ or $C_{\infty,\lambda}$ is violated, additional solutions of the difference equation (2.13) can be found via the same limit processes used to find additional solutions of the differential equation (2.5).

Remark 2.5. If n is a nonnegative integer, no restrictions on r, s and z are necessary for $\mathscr{E}_n(z,\lambda)$ to be well defined and for the results of Theorem 2.1 to hold. In this connection, care must be taken in certain limit processes which may arise. For example, suppose $r=0$, $s=1$, $\beta_1=1$. Then from (2.11), we have $A_1(n,\lambda)=-(2n+\lambda-2)(\lambda-1)\,[(n+\lambda-1)(2n+\lambda-3)]^{-1}$. If we want

$A_1(n,\lambda)$ for $n = 1$ and $\lambda = 1$, we must first set $n = 1$ and then let $\lambda \to 1$. Thus $A_1(n,\lambda) = -1$ for $n = 1$ and $A_1(n,\lambda) = 0$ for $n \neq 1$.

Consider the limit procedure (called a confluence with respect to λ)

$$\lim_{\lambda\to\infty} \mathscr{E}_n(z/\lambda, \lambda) = \lim_{\lambda\to\infty} {}_{r+3}F_s\left(\begin{matrix}-n, n+\lambda, \alpha_r, 1\\ \\ \beta_s\end{matrix}\middle| \frac{z}{\lambda}\right),$$

$$= {}_{r+2}F_s\left(\begin{matrix}-n, \alpha_r, 1\\ \\ \beta_s\end{matrix}\middle| z\right) = \mathscr{E}_n(z), \tag{2.33}$$

which is valid for $r + 2 \leqslant s$. Thus, results for the generalized Laguerre functions can be deduced from those for the generalized Jacobi functions. In fact, if we write down the conditions

$$\left.\begin{array}{l} r+2 \leqslant s \quad \text{or} \quad r+1 = s \quad \text{and} \quad |\arg(1-z)| < \pi, \\ \text{no two of the parameters, } \beta_h\,(h = 1,\ldots,s)\text{, differ by an} \\ \text{integer,} \end{array}\right\} C_0$$

$$\left.\begin{array}{l} r \geqslant s \quad \text{or} \quad r+1 = s \quad \text{and} \quad |\arg(1-1/z)| < \pi, \\ \text{no two of the parameters, } -n, \alpha_k\,(k = 1,\ldots,r)\text{, differ by an} \\ \text{integer,} \end{array}\right\} C_\infty$$

and let

$$\lim_{\lambda\to\infty} \Phi_n(z/\lambda, \lambda) = \Phi_n(z),$$

$$\Phi_n(z,\lambda) = \mathscr{K}_n(z,\lambda), \quad \mathscr{F}_{n,h}(z,\lambda)\ (h = 1,\ldots,s), \quad \mathscr{G}_{n,k}(z,\lambda)\ (k = 1,\ldots,r+1), \tag{2.34}$$

a limiting form of Theorem 2.1 and Corollary 2.1 is the following.

Corollary 2.2. *The functions $\mathscr{E}_n(z)$ and $\mathscr{K}_n(z)$ under the conditions on r, s and z in C_0 and C_∞, respectively, and the function*

$$\mathscr{E}(n,z) = \frac{\Gamma(\beta_s)}{\Gamma(-n)\,\Gamma(\alpha_r)}\, G^{1,r+2}_{r+2,s+1}\left(-z\middle|\begin{matrix}0, 1+n, 1-\alpha_r\\ \\ 0, 1-\beta_s\end{matrix}\right) \tag{2.35}$$

satisfy the difference equation

$$\Phi_n(z) + \sum_{m=1}^{\bar{t}} [A_m(n) + zB_m(n)]\,\Phi_{n-m}(z) = \frac{(\beta_s - 1)}{(n + \beta_s - 1)},$$

$$\bar{t} = \max(r+1, s), \tag{2.36}$$

where

$$A_m(n) = \frac{(n+1-m)_m(n-m-1+\beta_s)}{m!\,(n-1+\beta_s)}\ {}_{s+1}F_s\left(\begin{matrix}-m, n-m+\beta_s \\ \\ n-m-1+\beta_s\end{matrix}\middle| 1\right),$$

$$= \frac{(n+1-m)_m(-)^m}{m!\,(n-1+\beta_s)} \sum_{u=0}^{s-m} \frac{(s-u)!}{(s-m-u)!} B_{s-m-u}^{(-m)} S_u(n-m-1+\beta_s), \tag{2.37}$$

$$B_m(n) = \frac{(n+1-m)_m(n-m+\alpha_r)}{(m-1)!\,(n-1+\beta_s)}\ {}_{r+1}F_r\left(\begin{matrix}1-m, n-m+1+\alpha_r \\ \\ n-m+\alpha_r\end{matrix}\middle| 1\right),$$

$$= \frac{(n+1-m)_m(-)^{m-1}}{(m-1)!\,(n-1+\beta_s)} \sum_{u=0}^{r+1-m} \frac{(r-u)!}{(r+1-m-u)!} B_{r+1-m-u}^{(1-m)} S_u(n-m+\alpha_r), \tag{2.38}$$

where $B_k^{(a)}$ *is the generalized Bernoulli number defined in* (A.7), *and where the* $S_u(\rho_q)$ *are the symmetric polynomials defined implicitly by*

$$\prod_{i=1}^{q} (x+\rho_i) = \sum_{u=0}^{q} S_u(\rho_q)\, x^{q-u}. \tag{2.39}$$

In particular, $A_m(n) = 0$, $m \geqslant s+1$ *and* $B_m(n) = 0$, $m \geqslant r+2$. *In addition, the functions* $\mathscr{F}_{n,h}(z)$ $(h = 1, \ldots, s)$ and $\mathscr{G}_{n,k}(z)$ $(k = 1, \ldots, r+1)$ *under the conditions* C_0 *and* C_∞, *respectively, satisfy the difference equation*

$$\Phi_n(z) + \sum_{m=1}^{\bar{t}} [A_m(n) + zB_m(n)]\Phi_{n-m}(z) = 0. \tag{2.40}$$

Finally, if no α_k *is equal to any* β_h, *none of the above functions satisfies a non-trivial equation of the form specified of lower order than* $\bar{t}$.

Remark 2.6. The only part of Corollary 2.2 which does not follow directly from Theorem 2.1 is the last statement, which is actually concerned with the uniqueness of

$$A_m(n) = \lim_{\lambda\to\infty} A_m(n, \lambda), \qquad B_m(n) = \lim_{\lambda\to\infty} \frac{B_m(n,\lambda)}{\lambda}.$$

The uniqueness of the $A_m(n)$ and $B_m(n)$ follows from the limiting form of Lemma 2.1 when $\epsilon \to \infty$, i.e.,

LEMMA 2.2. *If* $P_q(x)$ *is a polynomial in* x *of degree* q,

$$P_q(x) = c\prod_{i=1}^{q} (x-\omega_i), \tag{2.41}$$

and t is an integer $\geqslant q$, *then* $P_q(x)$ *can be represented uniquely in the form*

$$P_q(x) = c \sum_{m=0}^{t} (x+\gamma)_m \bar{Q}_m, \tag{2.42}$$

$$\bar{Q}_m = \frac{(-)^q(\gamma+\omega_q)\,c}{m!}\ {}_{q+1}F_q\left(\begin{matrix}-m, 1+\gamma+\omega_q \\ \gamma+\omega_q\end{matrix}\middle| 1\right),$$

$$= \frac{(-)^{q-m}\,c}{m!} \sum_{u=0}^{q-m} \frac{(q-u)!}{(q-m-u)!} B_{q-m-u}^{(-m)} S_u(\gamma+\omega_q), \tag{2.43}$$

where $B_k^{(a)}$ *is the generalized Bernoulli number defined in* (A.7) *and where the* $S_u(\rho_q)$ *are the symmetric polynomials defined implicitly by* (2.39). *Note that* $\bar{Q}_m = 0$ *for* $m \geqslant q+1$. *If* $(\gamma+\omega_q)$ *is zero, limits must be taken in* (2.43).

Remark 2.7. The second lines of (2.37, 38, 43) follow by an application of Lemma A.2. in the Appendix.

Remarks similar to those following Corollary 2.1 can also be made for Corollary 2.2.

To illustrate the principal results of this section, we have for n a positive integer that

$$\mathscr{P}_n(z,\lambda) = {}_4F_3\left(\begin{matrix}-n, n+\lambda, \alpha_1, \alpha_2 \\ \beta_1, \beta_2, \beta_3\end{matrix}\middle| z\right) \tag{2.44}$$

satisfies the difference equation

$$\Phi_n(z,\lambda) + \sum_{m=1}^{4} [A_m(n,\lambda) + zB_m(n,\lambda)]\Phi_{n-m}(z,\lambda) = 0, \qquad B_4(n,\lambda) = 0, \tag{2.45}$$

where

$$A_1(n,\lambda) = \frac{(n-1)(2n+\lambda-2)_2(n-2+\beta)}{(n+\lambda-1)(2n+\lambda-5)(n-1+\beta)}\left\{1 - \frac{(2n+\lambda-5)(n)(n-1+\beta)}{(2n+\lambda-1)(n-1)(n-2+\beta)}\right\},$$

$$A_2(n,\lambda) = \frac{(n-2)_2(2n+\lambda-4)_4(n-3+\beta)}{2(n+\lambda-2)_2(2n+\lambda-6)_2(n-1+\beta)}\left\{1 - \frac{2(2n+\lambda-6)(n-1)(n-2+\beta)}{(2n+\lambda-3)(n-2)(n-3+\beta)} + \frac{(2n+\lambda-6)_2(n)(n-1+\beta)}{(2n+\lambda-3)_2(n-2)(n-3+\beta)}\right\},$$

$$A_3(n,\lambda)$$
$$= \frac{(n-2)_2(n+\lambda-4)(2n+\lambda-6)(2n+\lambda-3)_3(n+\lambda-3-\beta)}{(n+\lambda-3)_3(2n+\lambda-7)_3(n-1+\beta)}$$
$$\times \left\{1 - \frac{(2n+\lambda-7)(n+\lambda-3)(n+\lambda-2-\beta)}{(2n+\lambda-3)(n+\lambda-4)(n+\lambda-3-\beta)}\right\},$$

$$A_4(n,\lambda)$$
$$= \frac{(n-3)_3(n+\lambda-4)(2n+\lambda-8)(2n+\lambda-3)_3(n+\lambda-3-\beta)}{(n+\lambda-4)_4(2n+\lambda-8)_4(n-1+\beta)},\tag{2.46}$$

$$1 + \sum_{m=1}^{4} A_m(n,\lambda) = 0,$$

$$B_1(n,\lambda)$$
$$= \frac{(2n+\lambda-2)_2(n-1+\alpha)}{(n+\lambda-1)(n-1+\beta)},$$

$$B_2(n,\lambda)$$
$$= \frac{(n-1)(2n+\lambda-4)_4(n-2+\alpha)}{(n+\lambda-2)_2(2n+\lambda-5)(n-1+\beta)}\left\{1 - \frac{(2n+\lambda-5)(n-1+\alpha)}{(2n+\lambda-3)(n-2+\alpha)}\right\},$$

$$B_3(n,\lambda)$$
$$= \frac{(n-2)_2(2n+\lambda-3)_3(n+\lambda-3-\alpha)}{(n+\lambda-3)_3(2n+\lambda-5)(n-1+\beta)}.$$

Here $(n+u+\beta)$ is short for

$$\prod_{j=1}^{3} (n+u+\beta_j)$$

and $(n+u+\alpha)$ is short for

$$\prod_{j=1}^{2} (n+u+\alpha_j).$$

Similar recurrence formulae for any hypergeometric function of lower order than $\mathscr{P}_n(z,\lambda)$ can be found by taking limiting forms of (2.45) and (2.46). In particular, recurrence relationships for

$${}_4F_2\left(\begin{matrix}-n, n+\lambda, \alpha_1, \alpha_2\\ \beta_1, \beta_2\end{matrix}\middle| z\right); \qquad {}_3F_3\left(\begin{matrix}-n, \alpha_1, \alpha_2\\ \beta_1, \beta_2, \beta_3\end{matrix}\middle| z\right),$$

may be found by replacing z in (2.45) by $z\beta_3$; z/λ and letting β_3; $\lambda \to \infty$. If both numerator and denominator parameters are to be removed from $\mathscr{P}_n(z,\lambda)$, care must be taken to obtain the nontrivial recurrence relation of lowest order e.g., to obtain the recurrence relation for

$$ {}_3F_2\left(\begin{matrix}-n, n+\lambda, \alpha_1 \\ \beta_1, \beta_2\end{matrix}\middle| z\right) $$

from (2.45), we set $\alpha_2 = \beta_3 = n + \lambda + 1 - t$, $t = 4$, so that $A_4(n,\lambda) = B_3(n,\lambda) = 0$. Recurrence formulae for special cases of the above have been given by Fasenmeyer [*12*], and Rainville [*13*].

III. Recursion Formulae for the Numerator Polynomials in the Rational Approximations for the Generalized Hypergeometric Function

Suppose

$$ F(z) = \sum_{r=0}^{n-1} f_r z^r + R_n(z), \tag{3.1} $$

where f_r is independent of n and z. In (3.1), replace n by $k + 1 - a$, $a = 0$ or 1, multiply both sides by $A_{n,k}\gamma^k$, $A_{n,k} = 0$ if $k > n$, and sum from $k = 0$ to $k = n$. Then, just as in (1.9), (1.10), we obtain

$$ F(z)\, h_n(\gamma) = \psi_n(z,\gamma) + S_n(z,\gamma), \tag{3.2} $$

$$ h_n(\gamma) = \sum_{k=0}^{n} A_{n,k}\gamma^k, \tag{3.3} $$

$$ \psi_n(z,\gamma) = \sum_{k=a}^{n} \gamma^k \sum_{r=0}^{n-k} A_{n,r+k}\, f_r(z\gamma)^r, \tag{3.4} $$

which can be interpreted as giving a formal rational approximation to $F(z)$, i.e., $\psi_n(z,\gamma)/h_n(\gamma)$. As in (1.10), alternate expressions can be given for $\psi_n(z,\gamma)$. In particular, $\psi_n(z,\gamma)$ is a weighted sum of partial sums of $F(z)$. We now prove that if $h_n(\gamma)$ satisfies a linear, inhomogeneous recursion relation, then $\psi_n(z,\gamma)$ satisfies another linear, inhomogeneous recursion relation and the homogeneous portions of these recurrence relations are identical. This general result is embodied in the following theorem.

THEOREM 3.1. *Let $h_n(\gamma)$, $\psi_n(z,\gamma)$ be defined as above. Let there exist constants $K_m(n)$, $L_m(n)$ and $M(n)$ such that*

$$ \sum_{m=0}^{t} [K_m(n) + \gamma L_m(n)]\, h_{n-m}(\gamma) = M(n), $$

$$ K_0(n) = 1, \qquad L_0(n) = 0, \qquad n \geqslant t. \tag{3.5} $$

Then

$$Q_n(z,\gamma)=\sum_{m=0}^{t}[K_m(n)+\gamma L_m(n)]\,\psi_{n-m}(z,\gamma)$$

$$=\gamma^a\sum_{r=0}^{n-a}f_r(z\gamma)^r\sum_{m=0}^{t}K_m(n)\,A_{n-m,\,r+a}. \tag{3.6}$$

Further, if

$$h_n(\gamma)={}_{r+3}F_s\left(\begin{matrix}-n,n+\lambda,\alpha_r,1\\ \\ \beta_s\end{matrix}\middle|\,\gamma\right), \tag{3.7}$$

and (3.5) *is identified with* (2.10), *then*

$$Q_n(z,\gamma)=[-\gamma n\alpha_r]^a[\beta_s-1]^{1-a}\frac{(2n+\lambda-t)_t\,\Gamma(n+\lambda+a-t)}{(n+\beta_s-1)\,\Gamma(n+\lambda)}$$

$$\times\sum_{j=0}^{n-a}\frac{(a-n)_j(n+\lambda+a-t)_j(\alpha_r+a)_j\,f_j(z\gamma)_j}{(\beta_s+a-1)_j},$$

$$t=\max\,(r+2,s). \tag{3.8}$$

Proof. Combining the first expression of (3.6) with (3.4), we have

$$Q_n(z,\gamma)=\sum_{m=0}^{t}[K_m(n)+\gamma L_m(n)]\sum_{k=a}^{n-m}\gamma^k\sum_{r=0}^{n-m-k}A_{n-m,\,r+k}\,f_r(z\gamma)^r.$$

Observe that we can replace the upper limits of the r and k summation indices by $n-a$ and n, respectively, since $A_{n,k}=0$ if $k>n$. Thus, we can write

$$Q_n(z,\gamma)=\sum_{m=0}^{t}[K_m(n)+\gamma L_m(n)]\sum_{k=a}^{n}\gamma^k\sum_{r=0}^{n-a}A_{n-m,\,r+k}\,f_r(z\gamma)^r$$

$$=\gamma^a\sum_{r=0}^{n-a}f_r(z\gamma)^r\sum_{m=0}^{t}K_m(n)\,A_{n-m,\,r+a}$$

$$+\sum_{k=a+1}^{n}\gamma^k\sum_{r=0}^{n-a}f_r(z\gamma)^r\sum_{m=0}^{t}[K_m(n)\,A_{n-m,\,r+k}+L_m(n)\,A_{n-m,\,r+k-1}].$$

But in view of (3.3) and (3.5)

$$\sum_{m=0}^{t}[K_m(n)\,A_{n-m,\,j}+L_m(n)\,A_{n-m,\,j-1}]=0,$$

for j a positive integer. It follows that $Q_n(z,\gamma)$ is given by the second line of (3.6).

Next, we turn to the proof of (3.8). With $A_{n,k}$ defined by (3.7), and

$$C_{n,k} = \frac{(-n)_k(n+\lambda-t)_k(\alpha_r)_k}{(\beta_s)_k},$$

$$= \frac{(-n)_m(n+\lambda-t)_{t-m}}{(k-n)_m(k+n+\lambda-t)_{t-m}} A_{n-m,k},$$

it follows from (2.15, 18) that

$$\sum_{m=0}^{t} K_m(n) A_{n-m,k} = C_{n,k} X_t(k),$$

$$= \frac{(\beta_s-1)(n+\lambda)_n(-n)_k(n+\lambda-t)_k(\alpha_r)_k}{(n+\beta_s-1)(n+\lambda-t)_n(\beta_s-1)_k}. \tag{3.9}$$

Combining (3.9) with (3.6), we arrive at (3.8). This concludes the proof of the theorem.

The following corollary summarizes these results in a form convenient for applications.

COROLLARY 3.1. *If* $a = 0$ *or* 1,

$$h_n(\gamma) = {}_{f+q+3}F_{g+p+1}\left(\begin{matrix} -n, n+\lambda, \sigma_f, \beta_q - a, 1 \\ \beta+1, \rho_g, \alpha_p + 1 - a \end{matrix}\middle| \gamma\right), \tag{3.10}$$

$$\psi_n(z,\gamma) = \left[-\frac{\gamma n(n+\lambda)\,\sigma_f(\beta_q-1)}{(\beta+1)\,\rho_g\,\alpha_p}\right]^a$$
$$\times \sum_{k=0}^{n-a} \frac{(-n+a)_k(n+\lambda+a)_k(\sigma_f+a)_k(\alpha_p)_k(\gamma z)^k}{(\beta+1+a)_k(\rho_g+a)_k(1+\alpha_p)_k\,k!}$$
$$\times {}_{f+q+3}F_{g+p+1}\left(\begin{matrix} -n+k+a, n+\lambda+k+a, \sigma_f+k+a, \beta_q+k, 1 \\ \beta+1+k+a, \rho_g+k+a, 1+\alpha_p+k \end{matrix}\middle| \gamma\right), \tag{3.11}$$

and $t = \max\{f+q+2, g+p+1\}$,

$$K_m(n,\lambda)$$
$$= \frac{(n+1-m)_m(2n+\lambda-2m)_{2m}(n-m+\beta)(n-m+\rho_g-1)(n-m+\alpha_p-a)}{m!\,(n+\lambda-m)_m(2n+\lambda-t-m)_m(n+\beta)(n+\rho_g-1)(n+\alpha_p-a)}$$
$$\times {}_{g+p+3}F_{g+p+2}\left(\begin{matrix} -m, 2n+\lambda-t-m, n-m+\beta+1, n-m+\rho_g, n-m+\alpha_p+1-a \\ 2n+\lambda+1-2m, n-m+\beta, n-m+\rho_g-1, n-m+\alpha_p-a \end{matrix}\middle| 1\right),$$

$$= (-1)^{g+p+1} \frac{(n+1-m)_m(2n+\lambda-2m)(2n+\lambda-t+1)_{t-1}(n+\lambda-t-\beta)(n+\lambda-t+1-\rho_g)(n+\lambda-t-\alpha_p+a)}{(t-m)!\,(n+\lambda-m)_m(2n+\lambda-t-m)_m(n+\beta)(n+\rho_g-1)(n+\alpha_p-a)}$$

$$\times\ {}_{g+p+3}F_{g+p+2}\left(\begin{matrix}-t+m,\ 2n+\lambda-t-m,\ n+\lambda-t+1-\beta,\ n+\lambda-t+2-\rho_g,\ n+\lambda-t+1-\alpha_p+a \\ 2n+\lambda+1-t,\ n+\lambda-t-\beta,\ n+\lambda-t+1-\rho_g,\ n+\lambda-t-\alpha_p+a\end{matrix}\middle|\, 1\right), \tag{3.12}$$

$L_m(n, \lambda)$

$$= \frac{(n+1-m)_m(2n+\lambda-2m)_{2m}(n-m+\sigma_f)(n-m+\beta_q-a)}{(m-1)!\,(n+\lambda-m)_m(2n+\lambda-t-m+1)_{m-1}(n+\beta)(n+\rho_g-1)(n+\alpha_p-a)}$$

$$\times\ {}_{f+q+2}F_{f+q+1}\left(\begin{matrix}1-m,\ 2n+\lambda-t-m+1,\ n-m+1+\sigma_f,\ n-m+1+\beta_q-a \\ 2n+\lambda+1-2m,\ n-m+\sigma_f,\ n-m+\beta_q-a\end{matrix}\middle|\, 1\right),$$

$$= (-1)^{f+q} \frac{(n+1-m)_m(2n+\lambda-2m)(2n+\lambda-t+1)_{t-1}(n+\lambda-t+1-\sigma_f)(n+\lambda-t+1-\beta_q+a)}{(t-m-1)!\,(n+\lambda-m)_m(2n+\lambda-t-m+1)_{m-1}(n+\beta)(n+\rho_g-1)(n+\alpha_p-a)}$$

$$\times\ {}_{f+q+2}F_{f+q+1}\left(\begin{matrix}-t+m+1,\ 2n+\lambda-t-m+1,\ n+\lambda-t+2-\sigma_f,\ n+\lambda-t+2-\beta_q+a \\ 2n+\lambda+1-t,\ n+\lambda-t+1-\sigma_f,\ n+\lambda-t+1-\beta_q+a\end{matrix}\middle|\, 1\right), \tag{3.13}$$

then $\psi_n(z,\gamma)/h_n(\gamma)$ *is a formal rational approximation to*

$$ {}_pF_q\left(\begin{matrix}\alpha_p \\ \beta_q\end{matrix}\middle|\, z\right),$$

such that

$$\sum_{m=0}^{t} [K_m(n,\lambda) + \gamma L_m(n,\lambda)]\, h_{n-m}(\gamma) = \frac{\beta(\rho_g - 1)(\alpha_p - a)(n+\lambda)_n}{(n+\beta)(n+\rho_g - 1)(n+\alpha_p - a)(n+\lambda-t)_n}, \tag{3.14}$$

and

$$\sum_{m=0}^{t} [K_m(n,\lambda) + \gamma L_m(n,\lambda)]\, \psi_{n-m}(z,\gamma)$$

$$= [-\gamma n \sigma_f(\beta_q - 1)]^a[\beta(\rho_g - 1)\,\alpha_p]^{1-a} \frac{(2n+\lambda-t)_t\, \Gamma(n+\lambda+a-t)}{(n+\beta)(n+\alpha_p-a)(n+\rho_g-1)\,\Gamma(n+\lambda)}$$

$$\times\ {}_{f+2}F_{g+1}\left(\begin{matrix}-n+a,\ n+\lambda-t+a,\ \sigma_f+a \\ \beta+a,\ \rho_g-1+a\end{matrix}\middle|\, \gamma z\right). \tag{3.15}$$

The difference equations (3.14) *and* (3.15) *are homogeneous if*

$$\beta(\rho_g - 1)(\alpha_p - a) = 0 \qquad \text{and} \qquad [\sigma_f(\beta_q - 1)]^a = 0,$$

respectively. In the special case $\gamma z = 1$, $f = g = 0$, $\lambda = \alpha + \beta + 1$, *Eq.* (3.15) *is also homogeneous, if* $2 + a + \alpha - t$ *is a nonpositive integer.*

Remark 3.1. Note that the ${}_{f+2}F_{g+1}$ on the right-hand side of (3.15) is an extended Jacobi polynomial and so may be generated by application of the recursion formula developed in Section II.

Remark 3.2. Care must be taken in (3.15) if $a = 0$ and $\beta(\rho_g - 1)$ approaches zero. In particular, one must use the relation

$$\beta(\rho_g - 1)\,{}_{f+2}F_{g+1}\left(\begin{matrix}-n, n+\lambda-t, \sigma_f \\ \beta, \rho_g - 1\end{matrix}\middle|\gamma z\right)$$
$$= \beta(\rho_g - 1)$$
$$+ (-n)(n+\lambda-t)(\sigma_f)(\gamma z)\,{}_{f+3}F_{g+2}\left(\begin{matrix}1-n, 1+n+\lambda-t, 1+\sigma_f, 1 \\ 1+\beta, \rho_g, 2\end{matrix}\middle|\gamma z\right). \tag{3.16}$$

Remark 3.3. Should a numerator parameter σ be equal to a denominator parameter ρ in (3.10), (3.14) and (3.15) will in general reduce in length only if $\sigma = \rho = n + \lambda + 1 - t$. For particular numerical values of $\sigma = \rho\,(\neq n + \lambda + 1 - t)$, (3.14) and (3.15), though still valid, are no longer the desired recursion formulae of shortest length.

Remark 3.4. Confluent forms of Theorem 3.1 and Corollary 3.1 are easily found by replacing γ by γ/λ, and letting $\lambda \to +\infty$.

Remark 3.5. An alternate technique for the evaluation of $\psi_n(z, \gamma)$ as defined by (3.11) is the following: if

$$V_{n,k}(\gamma) = {}_{r+3}F_s\left(\begin{matrix}-n+a+k, n+\lambda+a+k, \theta_r+k, 1 \\ \omega_s + k\end{matrix}\middle|\gamma\right), \tag{3.17}$$

then

$$V_{n,k-1}(\gamma) = 1 + \gamma\frac{(-n+a-1+k)(n+\lambda+a-1+k)(\theta_r-1+k)}{(\omega_s-1+k)} V_{n,k}(\gamma),$$
$$V_{n,n-a}(\gamma) = 1. \tag{3.18}$$

IV. Recursion Formulae for Particular G-Functions

In this section, we obtain a linear, homogeneous difference equation for the G-functions

$$\mathscr{V}(n, \omega, \lambda) = G^{m,l}_{p+1,q+2}\left(\omega\middle|\begin{matrix}a_{p+1} \\ b_q, n, -n-\lambda\end{matrix}\right),$$
$$0 \leqslant m \leqslant q, \qquad 0 \leqslant l \leqslant p+1. \tag{4.1}$$

Clearly, $\mathscr{V}(n,\omega,\lambda)$ includes those G-functions occurring in (1.16) as coefficients of the generalized Jacobi polynomials. As in Section II, properly normalized bases of

$$\{(-1)^{p-m-l}\,\omega(\delta+1-a_{p+1})+(\delta-b_q)(\delta-n)(\delta+n+\lambda)\}\,Y(\omega)=0,$$

$$\delta=\omega\frac{d}{d\omega}, \tag{4.2}$$

[an equation satisfied by $\mathscr{V}(n,\omega,\lambda)$], also satisfy the above-mentioned difference equation for $\mathscr{V}(n,\omega,\lambda)$. For example, if $p>q+1$, the functions,

$$\mathscr{U}_h(n,\omega,\lambda)=G^{1,\,p+1}_{p+1,\,q+2}\left(\omega e^{-i\pi(m+l-p)}\,\middle|\,\begin{matrix}a_{p+1}\\ b_h,b_1,\ldots,b_{h-1},b_{h+1},\ldots,b_q,n,-n-\lambda\end{matrix}\right),$$

$$h=1,\ldots,q,$$

$$\mathscr{U}_{q+1}(n,\omega,\lambda)=e^{i\phi n}\,G^{1,\,p+1}_{p+1,\,q+2}\left(\omega e^{-i\pi(m+l-p)}\,\middle|\,\begin{matrix}a_{p+1}\\ n,b_q,-n-\lambda\end{matrix}\right), \tag{4.3}$$

$$\mathscr{U}_{q+2}(n,\omega,\lambda)=e^{i\phi n}\,G^{1,\,p+1}_{p+1,\,q+2}\left(\omega e^{-i\pi(m+l-p)}\,\middle|\,\begin{matrix}a_{p+1}\\ -n-\lambda,b_q,n\end{matrix}\right),$$

with $e^{i\phi}=-1$, and

$$\mathscr{S}_k(n,\omega,\lambda)=G^{0,\,p+1}_{p+1,\,q+2}\left(\omega e^{i\pi(2k+m+l-p-1)}\,\middle|\,\begin{matrix}a_{p+1}\\ b_q,n,-n-\lambda\end{matrix}\right),$$

$$k=1,\ldots,p-(q+1), \tag{4.4}$$

form the desired basis, normalized with respect to n, in a proper sector of the irregular singular point $\omega=0$. Alternately, if $q+1>p$, the functions

$$\mathscr{H}_k(n,\omega,\lambda)=G^{q+2,\,1}_{p+1,\,q+2}\left(\omega e^{i\pi(q-m-l+1)}\,\middle|\,\begin{matrix}a_k,a_1,\ldots,a_{k-1},a_{k+1},\ldots,a_{p+1}\\ b_q,n,-n-\lambda\end{matrix}\right),$$

$$k=1,\ldots,p+1, \tag{4.5}$$

$$\mathscr{W}_h(n,\omega,\lambda)=G^{q+2,\,0}_{p+1,\,q+2}\left(\omega e^{-i\pi(2h+m+l-q)}\,\middle|\,\begin{matrix}a_{p+1}\\ b_q,-n,n+\lambda\end{matrix}\right),$$

$$h=1,\ldots,q+1-p, \tag{4.6}$$

form the desired basis, normalized with respect to n, in a proper sector of the irregular singular point $\omega = \infty$. We tacitly assume that all of the above functions are well defined.

We now state our main result of this section.

THEOREM 4.1. *Provided the functions $\mathscr{V}(n,\omega,\lambda)$, $\mathscr{U}_h(n,\omega,\lambda)$, $\mathscr{S}_k(n,\omega,\lambda)$, $\mathscr{H}_k(n,\omega,\lambda)$ or $\mathscr{W}_h(n,\omega,\lambda)$ are well defined, they satisfy the difference equation*

$$\Psi(n,\omega,\lambda) + \sum_{j=1}^{t} [C_j(n,\lambda) + \omega^{-1} D_j(n,\lambda)]\, \Psi(n+j,\omega,\lambda) = 0,$$

$$t = \max(q+2, p+1), \qquad D_t(n,\lambda) = 0, \tag{4.7}$$

where

$$\begin{aligned} C_j(n,\lambda) &= \frac{(-1)^j(2n+\lambda+2j)(2n+\lambda)_j}{j!\,(2n+\lambda)}\, {}_{p+3}F_{p+2}\left(\begin{matrix} -j, j+2n+\lambda, n+2-a_{p+1} \\ 2n+\lambda+t+1, n+1-a_{p+1} \end{matrix}\middle| 1\right), \\ &= \frac{(-1)^{j+p+1}(2n+\lambda+1)_t(n+\lambda+j-1+a_{p+1})}{(t-j)!\,(2n+\lambda+j)_j(n+1-a_{p+1})} \\ &\quad \times {}_{p+3}F_{p+2}\left(\begin{matrix} j-t, 2n+\lambda+j, n+\lambda+j+a_{p+1} \\ 2n+\lambda+2j+1, n+\lambda+j-1+a_{p+1} \end{matrix}\middle| 1\right), \end{aligned} \tag{4.8}$$

$$\begin{aligned} D_j(n,\lambda) &= \frac{(-1)^{j+p+m+l+1}(2n+\lambda+2j)(2n+\lambda+1)_j(n+1-b_q)}{(j-1)!\,(n+1-a_{p+1})} \\ &\quad \times {}_{q+2}F_{q+1}\left(\begin{matrix} -j+1, 2n+\lambda+j+1, n+2-b_q \\ 2n+\lambda+t+1, n+1-b_q \end{matrix}\middle| 1\right), \\ &= \frac{(-1)^{j+p+m+l+q+1}(2n+\lambda+j)(2n+\lambda+1)_t(n+\lambda+j+b_q)}{(t-j-1)!\,(2n+\lambda+j)_j(n+1-a_{p+1})} \\ &\quad \times {}_{q+2}F_{q+1}\left(\begin{matrix} j+1-t, 2n+\lambda+j+1, n+\lambda+j+1+b_q \\ 2n+\lambda+2j+1, n+\lambda+j+b_q \end{matrix}\middle| 1\right). \end{aligned} \tag{4.9}$$

Moreover, if no a_k is equal to any b_h, none of the above functions satisfy a non-trivial equation of the form specified of lower order than t.

Proof. Tentatively, we assume that no a_k is equal to any b_h and that ω is sufficiently restricted for the following manipulations to be valid. Consider the function $\mathscr{V}(n,\omega,\lambda)$. It follows from the integral representation for $\mathscr{V}(n,\omega,\lambda)$ [see (1.3)], that, for $C_0(n,\lambda)=1$,

$$\begin{aligned}
&\mathscr{Q}(n,\omega,\lambda)\\
&=\sum_{j=0}^{t}[C_j(n,\lambda)+\omega^{-1}D_j(n,\lambda)]\mathscr{V}(n+j,\omega,\lambda),\\
&=\frac{(-1)^t}{2\pi i}\int_L\frac{\prod_{k=1}^{m}\Gamma(b_k-v)\prod_{k=1}^{l}\Gamma(1-a_k+v)\,\omega^v\sum_{j=0}^{t}(n-v)_j(-n-\lambda-t-v)_{t-j}C_j(n,\lambda)}{\prod_{k=m+1}^{q}\Gamma(1-b_k+v)\prod_{k=l+1}^{p+1}\Gamma(a_k-v)\,\Gamma(v+1-n)\,\Gamma(v+n+\lambda+t+1)}\,dv\\
&+\frac{(-1)^t}{2\pi i}\int_{L-1}\frac{\prod_{k=1}^{m}\Gamma(b_k-1-v)\prod_{k=1}^{l}\Gamma(2-a_k+v)\,\omega^v\sum_{j=0}^{t}(n-1-v)_j(-n-\lambda-t-1-v)_{t-j}D_j(n,\lambda)}{\prod_{k=m+1}^{q}\Gamma(2-b_k+v)\prod_{k=l+1}^{p+1}\Gamma(a_k-1-v)\,\Gamma(v+2-n)\,\Gamma(v+n+\lambda+t+2)}\,dv,
\end{aligned}\tag{4.10}$$

where the contour L, independent of j, runs parallel to the imaginary axis, and is indented to separate the poles of $\Gamma(b_h-v)$ $(h=1,\ldots,m)$ from the poles of $\Gamma(1-a_k+v)$ $(k=1,\ldots,l)$. Let $R(v)$ denote the integrand of the first integral in (4.10). Then, moving the contour of integration of the first integral in (4.10) one unit to the left, we obtain

$$\begin{aligned}
&\mathscr{Q}(n,\omega,\lambda)\\
&=\frac{(-1)^{m+t}}{2\pi i}\int_{L-1}\frac{\prod_{k=1}^{m}\Gamma(b_k-1-v)\prod_{k=1}^{l}(1-a_k+v)\,\omega^vP(v)}{\prod_{k=m+1}^{q}\Gamma(2-b_k+v)\prod_{k=l+1}^{p+1}\Gamma(a_k-v)\,\Gamma(v+2-n)\,\Gamma(v+n+\lambda+t+2)}\,dv+R
\end{aligned}\tag{4.11}$$

where

$$\begin{aligned}
P(v)=(v+1-n)(v+n+\lambda+t+1)&\prod_{h=1}^{q}(v+1-b_h)\\
&\times\sum_{j=0}^{t}(n-v)_j(-n-\lambda-t-v)_{t-j}C_j(n,\lambda)\\
&-(-1)^{p+l+m}\prod_{k=1}^{p+1}(v+1-a_k)\\
&\times\sum_{j=0}^{t}(n-1-v)_j(-n-\lambda-t-1-v)_{t-j}D_j(n,\lambda),
\end{aligned}$$

and R is the sum of the residues due to those poles of $R(v)$ which lie between L and $L-1$, if any.

We now determine the $C_j(n,\lambda)$, $D_j(n,\lambda)$ by requiring that $P(v)=0$ for all v, so that $\mathscr{Q}(n,\omega,\lambda)$ reduces to just R. Then, under the assumptions that

$t = \max(q+2, p+1)$ and $a_k \neq b_h$, it follows, just as in the proof of Theorem 2.1, that there exists a constant C such that

$$\begin{aligned}
&\sum_{j=0}^{t} (n-v)_j(-n-\lambda-t-v)_{t-j}\, C_j(n,\lambda) \\
&= C(-1)^{p+l+m} \prod_{k=1}^{p+1} (v+1-a_k), \\
&\sum_{j=0}^{t} (n-1-v)_j(-n-\lambda-t-1-v)_{t-j}\, D_j(n,\lambda) \\
&= C(v+1-n)(v+n+\lambda+t+1) \prod_{h=1}^{q} (v+1-b_h).
\end{aligned} \tag{4.12}$$

Setting $v = n$ in the first line of (4.12), we find

$$C = (-1)^{t+p+l+m} \frac{(2n+\lambda+1)_t}{(n+1-a_{p+1})}. \tag{4.13}$$

The values of $C_j(n,\lambda)$, $D_j(n,\lambda)$ given in (4.8), (4.9) then follow directly from (4.12, 13) and Lemma 2.1. It also follows from (4.12) that

$$R(v) = C(-1)^{m+1} \times \frac{\prod_{k=1}^{m} \Gamma(b_k - v) \prod_{k=1}^{l} \Gamma(2-a_k+v)\, \omega^v}{\prod_{k=m+1}^{q} \Gamma(1-b_k+v) \prod_{k=l+1}^{p+1} \Gamma(a_k-1-v)\, \Gamma(v+1-n)\, \Gamma(v+n+\lambda+t+1)},$$

which clearly has no poles between L and $L-1$. Thus $\mathscr{Q}(n,\omega,\lambda) = R = 0$. Under our assumption that no a_k is equal to any b_h, the $C_j(n,\lambda)$, $D_j(n,\lambda)$ are unique, which implies that $\mathscr{V}(n,\omega,\lambda)$ does not satisfy a nontrivial equation of the form specified of lower order than t. As before, the tentative assumptions on a_k, b_h and ω can be relaxed completely by an appeal to continuity. Similarly, the other functions of the theorem can be shown to satisfy (4.7). This completes the proof.

Remark 4.1. The functions $e^{in\phi}\, \mathscr{V}(n,\omega,\lambda)$ with $m = q+1$, $e^{i\phi} = -1$, and $\mathscr{V}(n,\omega,\lambda)$ with $m = q+2$ also satisfy (4.7).

Remark 4.2. Confluent limits can be taken in Theorem 4.1. In particular, recursion formulae for

$$\begin{aligned}
\mathscr{V}(n,\omega) &= \lim_{\lambda\to\infty} \Gamma(n+\lambda+1)\, \mathscr{V}(n,\omega\lambda,\lambda), \\
&= G^{m,l}_{p+1,q+2}\left(\omega \,\middle|\, \begin{matrix} a_{p+1} \\ b_q, n \end{matrix}\right), \\
&0 \leqslant l \leqslant p+1, \qquad 0 \leqslant m \leqslant q,
\end{aligned} \tag{4.14}$$

can be deduced from (4.7). The $\mathscr{V}(n, \omega)$ occur as coefficients in the generalized Laguerre expansion (1.17).

Remark 4.3. To expand an arbitrary hypergeometric function ${}_{p+1}F_q(\omega z)$, $p \leqslant q$, in a series of extended Jacobi polynomials, it follows from (1.18) that it is sufficient to consider

$$\mathscr{C}_n(\omega, \lambda) = \frac{(\alpha_{p+1})_n \omega^n}{(\beta_q)_n (n+\lambda)_n} {}_{p+1}F_{q+1}\left(\left.\begin{matrix} n+\alpha_{p+1} \\ n+\beta_q, 2n+\lambda+1 \end{matrix}\right| \omega\right),$$
$$= \frac{\Gamma(\beta_q)(2n+\lambda)\Gamma(n+\lambda)}{\Gamma(\alpha_{p+1})} (-1)^n G^{1,p+1}_{p+1,q+2}\left(-\omega \left| \begin{matrix} 1-\alpha_{p+1} \\ n, 1-\beta_q, -n-\lambda \end{matrix}\right.\right), \tag{4.15}$$

n a nonnegative integer, $p \leqslant q$, or $p = q+1$ and $|\arg(1-z)| < \pi$. Comparing (4.3) and (4.15), we see that

$$\mathscr{C}_n(\omega, \lambda) = \frac{\Gamma(\beta_q)(2n+\lambda)\Gamma(n+\lambda)}{\Gamma(\alpha_{p+1})} \mathscr{U}_{q+1}(n, \omega, \lambda) \tag{4.16}$$

with $l = p+1$, $m = 0$, $1-\alpha_{p+1} = a_{p+1}$ and $1-\beta_q = b_q$. Thus, recursion formulae for $\mathscr{C}_n(\omega, \lambda)$ can be deduced from Theorem 4.1. In particular,

$$\mathscr{Q}_n(\omega, \lambda) = \frac{(\alpha_1)_n(\alpha_2)_n(\alpha_3)_n(\alpha_4)_n}{(\beta_1)_n(\beta_2)_n(n+\lambda)_n} \omega^n {}_4F_3\left(\left.\begin{matrix} n+\alpha_1, n+\alpha_2, n+\alpha_3, n+\alpha_4 \\ n+\beta_1, n+\beta_2, 2n+\lambda+1 \end{matrix}\right| \omega\right) \tag{4.17}$$

satisfies the difference equation

$$\Phi_n(\omega, \lambda) + \sum_{j=1}^{4} [E_j(n, \lambda) + \omega^{-1} F_j(n, \lambda)] \Phi_{n+j}(\omega, \lambda) = 0, \qquad F_4(n, \lambda) = 0, \tag{4.18}$$

where

$$E_1(n, \lambda) = -\frac{(2n+\lambda)}{(n+\lambda)}\left\{1 - \frac{(2n+\lambda+1)(n+\alpha+1)}{(2n+\lambda+5)(n+\alpha)}\right\},$$

$$E_2(n, \lambda) = \frac{(2n+\lambda)_2}{2(n+\lambda)_2}\left\{1 - \frac{2(2n+\lambda+2)(n+\alpha+1)}{(2n+\lambda+5)(n+\alpha)} + \frac{(2n+\lambda+2)_2(n+\alpha+2)}{(2n+\lambda+5)_2(n+\alpha)}\right\},$$

$$E_3(n, \lambda) = -\frac{(2n+\lambda)_3(n+\lambda+3-\alpha)}{(n+\lambda)_3(2n+\lambda+5)_2(n+\alpha)}\left\{1 - \frac{(2n+\lambda+3)(n+\lambda+4-\alpha)}{(2n+\lambda+7)(n+\lambda+3-\alpha)}\right\},$$

$$E_4(n, \lambda) = \frac{(2n+\lambda)_4(n+\lambda+4-\alpha)}{(n+\lambda)_4(2n+\lambda+5)_4(n+\alpha)}, \tag{4.19}$$

$$F_1(n,\lambda) = -\frac{(2n+\lambda)_2(n+\beta)}{(n+\lambda)(n+\alpha)},$$

$$F_2(n,\lambda) = \frac{(2n+\lambda)_3(n+\beta)}{(n+\lambda)_2(n+\alpha)}\left\{1 - \frac{(2n+\lambda+3)(n+\beta+1)}{(2n+\lambda+5)(n+\beta)}\right\},$$

$$F_3(n,\lambda) = -\frac{(2n+\lambda)_4(n+\lambda+4-\beta)}{(n+\lambda)_3(2n+\lambda+5)_2(n+\alpha)}.$$

Here, $(n+u+\alpha)$ is short for

$$\prod_{j=1}^{4}(n+u+\alpha_j),$$

and $(n+u+\beta)$ is short for

$$\prod_{j=1}^{2}(n+u+\beta_j).$$

Similar recurrence formulae for any hypergeometric function of lower order than $\mathcal{Q}_n(z,\lambda)$ can be found by taking limiting forms of (4.18) and (4.19).

Appendix

LEMMA A.1.

$$ {}_4F_3\left(\begin{matrix}-n, \beta+1, 1, z+2\beta \\ n+2\beta+1, \beta, 1-z\end{matrix}\middle| 1\right) = \frac{z(n+2\beta)}{2\beta(z-n)},$$

$$n = 0, 1, 2, \ldots; \qquad \beta(z-n) \neq 0. \tag{A.1}$$

Proof. Let $V(z)$ equal $(-z)^{-1}$ times the hypergeometric function appearing on the left-hand side of (A.1). Clearly $V(z)$ is a rational function of z, with the degree of the denominator polynomial one greater than that of the numerator polynomial. Moreover, as the poles of $V(z)$ are simple and located at $0, 1, \ldots, n$, we can write

$$V(z) = \sum_{j=0}^{n} \frac{q_j}{z-i},$$

$$q_j = \frac{(-1)^{j-1}(-n)_j(\beta+1)_j(2\beta+j)_j}{j!\,(n+2\beta+1)_j(\beta)_j}\,{}_3F_2\left(\begin{matrix}-n+j, \beta+1+j, 2j+2\beta \\ n+2\beta+1+j, \beta+j\end{matrix}\middle| 1\right). \tag{A.2}$$

Then, making use of the relation

$$
{}_3F_2\left(\begin{matrix}-n+j,\beta+1+j,2j+2\beta\\ n+2\beta+1+j,\beta+j\end{matrix}\middle|1\right)
$$

$$
={}_2F_1\left(\begin{matrix}-n+j,2j+2\beta\\ n+2\beta+1+j\end{matrix}\middle|1\right)-\frac{2(n-j)}{(n+2\beta+1+j)}\,{}_2F_1\left(\begin{matrix}-n+j+1,2j+2\beta+1\\ n+2\beta+2+j\end{matrix}\middle|1\right), \tag{A.3}
$$

together with Gauss' theorem [6] for summing a ${}_2F_1$ of unit argument, we find

$$
\begin{aligned}
q_j &= (-1)\frac{(n+2\beta)}{2\beta}, \qquad j=n,\\
&= 0, \qquad j=0,1,\ldots,n-1,
\end{aligned} \tag{A.4}
$$

which proves (A.1).

LEMMA A.2.

$$
(-)^m(a_p)_1\;{}_{p+1}F_p\left(\begin{matrix}-m,1+a_p\\ a_p\end{matrix}\middle|1\right)=\sum_{j=0}^{p-m}\frac{(j+m)!}{j!}B_j^{(-m)}S_{p-m-j}(a_p), \tag{A.5}
$$

where the $S_r(a_p)$ are the symmetric polynomials defined by

$$
\prod_{r=1}^{p}(x+a_r)=\sum_{r=0}^{p}S_r(a_p)\,x^{p-r}, \tag{A.6}
$$

and the $B_j^{(-m)}$ are the generalized Bernoulli numbers defined by

$$
\left(\frac{e^t-1}{t}\right)^m=\sum_{j=0}^{\infty}\frac{t^j}{j!}B_j^{(-m)}, \qquad B_0^{(-m)}=1. \tag{A.7}
$$

If $(a_p)_1$ is zero, limits must be taken in (A.5).

Proof. Let $\delta = xD = x(d/dx)$. It follows from the simple operator equation $(\delta)(\delta-1)\cdots(\delta-n+1)=x^2\,D^n$, and the finite difference formula, see [*14*, p. 150, Eq. (90)],

$$
x^k=\sum_{r=0}^{k}\binom{k}{r}B_{k-r}^{(-r)}\,x(x-1)\cdots(x-r+1), \qquad k=0,1,\ldots, \tag{A.8}
$$

with x replaced by δ, that

$$
\delta^k=\sum_{r=0}^{k}\binom{k}{r}B_{k-r}^{(-r)}\,x^r\,D^r, \tag{A.9}
$$

or

$$\prod_{j=1}^{p} (\delta + a_j) = \sum_{m=0}^{p} x^m D^m \sum_{j=0}^{p-m} \binom{m+j}{m} B_j^{(-m)} S_{p-m-j}(a_p). \tag{A.10}$$

Now letting (A.10) operate on x^ρ, and making use of the operator equation $(\delta + \sigma)x^\rho = x^\rho(\rho + \sigma)$, we obtain

$$\prod_{j=1}^{p} (x + a_i) = \sum_{m=0}^{p} x(x-1)\cdots(x-m+1) \sum_{j=0}^{p-m} \binom{m+j}{m} B_j^{(-m)} S_{p-m-j}(a_p), \tag{A.11}$$

or

$$\sum_{j=0}^{p-m} \binom{m+j}{m} B_j^{(-m)} S_{p-m-j}(a_p) = \frac{1}{m!} \Delta^m \left\{ \prod_{j=1}^{p} (x + a_j) \right\} \Bigg|_{x=0}, \tag{A.12}$$

where Δ is the forward difference operator with respect to x. Since

$$\frac{1}{m!} \Delta^m \left\{ \prod_{j=1}^{p} (x + a_j) \right\} \Bigg|_{x=0} = \frac{(-)^m}{m!} \sum_{r=0}^{m} \frac{(-m)_r}{r!} \prod_{j=1}^{p} (r + a_j), \tag{A.13}$$

(A.12) reduces to (A.5).

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